Determine if each statement is true or false.

1. If c is a critical point for a function f, then f has a local maximum or local minimum at c.

   (a) True
   (b) False

   Consider, for example, $f(x)=x^3$. Then $c=0$ is critical, since $f'(0)=0$, but there is not a maximum or minimum

2. If f is continuous on the closed interval $[a, b]$ and $f(a) = f(b)$ then there is some c in (a, b) such that $f'(c) = 0$.

   (a) True
   (b) False

   This is almost true by Rolles, but we don't know that f is differentiable on (a,b). Hence, counterexample exist, like $f(x)=|x|$ on $[-1,1]$.

3. $\lim_{x\to 10} \frac{x+1}{x^2-4} = \lim_{x\to 10} \frac{1}{2x} = \frac{1}{20}$

   (a) True
   (b) False

   Not indeterminate, so can't use L'Hospital's Rule. (In fact by direct substitution the limit is $\frac{11}{96}$.)

4. The Mean Value Theorem says that for $f(x) = |x|$ there must be a c in the interval $[-1, 1]$ such that $f'(c) = 0$.

   (a) True
   (b) False

   The MVT does not apply for $f(x)=|x|$ on $[-1,1]$ since f is not differentiable at $x=0$.

5. If $f'(x) > 0$ for $0 < x < 10$, then x is increasing on $(1, 10)$.

   (a) True
   (b) False

   (Note that $f'(x)>0$ means in particular f is differentiable, hence continuous, on $(1,10)$.)

6. If $f(x) = x^4 + 1$, then f has an inflection point at $x = 0$.

(a) True

(b) False

$f''(0) = 0$, but f is concave up everywhere so it has no inflection points.

7. There exists a function f such that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all x.

(a) True

(b) False

For example, $f(x) = e^{-x}$.

(Note that any combination is possible!)

8. If f has an absolute minimum at $x = c$ then f has a local minimum at $x = c$.

(a) True

(b) False

If the value is the smallest of any X then it is the smallest for nearby X.

9. L'Hospital's Rule says that for an indeterminate form

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} D_x \left(\frac{f(x)}{g(x)} \right)$$

Not equal!

(a) True

(b) False

Really: $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$ (for an ind. form.)

10. The linear function whose graph is the tangent line (the *linearization* of f at a) is given by

$$L(x) = f(a) + f'(a)(x - a).$$

(a) True

(b) False

11. If $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$ then $\lim_{x\to a} \frac{f(x)}{g(x)}$

(a) $= 0$

(b) $= 1$

(c) is ∞

(d) is undefined (but not necessarily infinite)

(e) might be any of the above, but we cannot know for certain based on the information

This is an indeterminate form, which might be any of the first four choices but is not necessarily any.

12. What is $\lim_{x\to -2} \frac{x^3+8}{x+2}$? $\underset{\text{"}\frac{0}{0}\text{"}}{}\overset{LR}{=} \lim_{x\to -2} \frac{3x^2}{1} \overset{\text{Dir.}}{\underset{\text{Subst}}{=}} \frac{12}{1} = 12$

(a) -6

(b) -2

(c) -1

(d) 0

(e) 2

(f) 12

(g) 8

(h) 6

(i) none of the above

(Or, you can solve by factoring:

$$= \lim_{x\to -2} \frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{x+2}}$$

$$= \lim_{x\to -2} (x^2-2x+4)$$

$$= 4+4+4 = 12.\ )$$

[Numerical check: $\frac{-1.999^3+8}{-1.999+2} = 11.99401.$]

13. What are the absolute maximum and minimum values of $f(x) = 4x + \frac{1}{x}$ on $[\frac{1}{4}, 4]$?

(a) 5 and 4
(b) 5 and -4
(c) 16.25 and 4
(d) 16.25 and -4
(e) 16.25 and 5
(f) 5 and 3.75
(g) none of the above

$f'(x) = 4 - \frac{1}{x^2} = 0 \rightarrow 4 = \frac{1}{x^2}$

$x^2 = \frac{1}{4} \rightarrow x = \frac{1}{2}$ or $\cancel{-\frac{1}{2}}$

Not in $[\frac{1}{4}, 4]$

Candidates: $x = \frac{1}{4}, x = \frac{1}{2}, x = 4$

$\underbrace{f(\tfrac{1}{4}) = 5, \ f(\tfrac{1}{2}) = 4}_{\text{minimum}}, \ \underbrace{f(4) = 16 + \tfrac{1}{4}}_{\text{maximum}}$

14. Use linear approximation to estimate $(999)^{\frac{1}{3}}$.

(a) $\frac{299}{300} \approx (0.9967)$
(b) $\frac{3001}{300} \approx (10.0033)$
(c) $\frac{2999}{300} \approx (9.9967)$
(d) $\frac{2199}{200} \approx (10.995)$
(e) $\frac{999}{100} \approx (9.990)$
(f) none of the above

$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ $\quad f(1000) = \sqrt[3]{1000} = 10$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ $\quad f'(1000) = \frac{1}{3} \cdot \frac{1}{10^2} = \frac{1}{300}$

$a = 1000$

By linear approximation:

$$f(999) = (999)^{1/3} \approx 10 + \frac{1}{300}(999 - 1000)$$

$$= 10 - \frac{1}{300} = \frac{3000}{300} - \frac{1}{300}$$

$$= \frac{2999}{300} \approx 9.9967$$

✓ Continuous & differentiable

15. Consider the function $f(x) = x^2 + 4x$ on the interval [0,2]. What is the point c guaranteed by the Mean Value Theorem.

(a) $x = -3$
(b) $x = -2$
(c) $x = -1$
(d) $x = 0$
(e) $x = 1$
(f) $x = 2$
(g) $x = 3$
(h) $x = 6$
(i) none of the above

$f(0)=0$ $f(2) = 2^2+4\cdot 2 = 12$, so there is a c with

$f'(c) = \dfrac{12-0}{2-0}$

$f'(x) = 2x+4$ $\rightarrow$ $2c+4 = 6$

$2c = 2$

$\boxed{c=1}$

16. The function $f(x) = \frac{e^x}{x-1}$ has exactly one critical number. Find it and determine if it corresponds to a local maximum, local minimum, or neither.

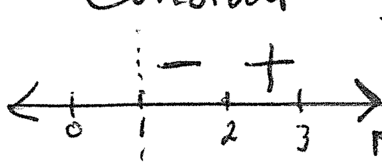
(a) 0, local min
(b) 1, local min
(c) 2, local min
(d) 0, local max
(e) 1, local max
(f) 2, local max
(g) 0, neither
(h) 1, neither
(i) 2, neither
(j) none of the above

$f'(x) = \dfrac{(x-1)e^x - e^x\cdot 1}{(x-1)^2} = \dfrac{xe^x - e^x - e^x}{(x-1)^2}$

so $f'(x) = \dfrac{e^x(x-2)}{(x-1)^2}$

$f'(x) = 0$ if $e^x(x-2) = 0 \Rightarrow \boxed{x=2}$

Consider $\dfrac{e^x(x-2)}{(x-1)^2}$ $\rightarrow$ $x>2$ $\frac{+\cdot+}{+} = +$; $\rightarrow$ $x<2$ $\frac{+\cdot-}{+} = -$



$\boxed{\text{By F.D.T, local min}}$

17. If dx=0.1, x=2, and $f(x) = x^3 - 7x$, find dy.

(a) $\frac{1}{7}$
(b) $\frac{1}{6}$
(c) $\frac{1}{5}$
(d) $\frac{1}{4}$
(e) $\frac{1}{3}$
(f) $\frac{1}{2}$
(g) none of the above

$$dy = (3x^2 - 7)dx = (3 \cdot 2^2 - 7)(0.1)$$
$$= 5 \cdot \frac{1}{10} = \frac{1}{2}$$

18. Find $\lim_{x \to \pi} \frac{\cos(x) + 1}{\sin(x)}$ $\overset{LR}{=} \lim_{x \to \pi} \frac{-\sin x}{\cos x} = \frac{0}{-1} = 0$

"0" / 0

(a) $x = -3$
(b) $x = -2$
(c) $x = -1$
(d) $x = 0$
(e) $x = 1$
(f) $x = 2$
(g) $x = 3$
(h) $x = 6$
(i) none of the above

19. Compute $\lim_{x\to 1^+} \frac{1}{\ln x} - \frac{1}{x-1}$. $= \lim_{x\to 1^+} \frac{(x-1)-\ln x}{\ln x(x-1)} \leftarrow$ "$\frac{0}{0}$"

(a) $\sin(x)$
(b) $\cos(x)$
(c) $-\sin(x)$
(d) $-\cos(x)$
(e) none of the above

$\overset{LR}{=} \lim_{x\to 1^+} \frac{1-\frac{1}{x}}{\frac{1}{x}(x-1)+\ln x}$

$\overset{(alg)}{=} \lim_{x\to 1^+} \frac{x-1}{x-1+x\ln x}$ Still "$\frac{0}{0}$"

$\overset{LR}{=} \lim_{x\to 1^+} \frac{1}{1+\ln x+1} \leftarrow$ Not indeterminate

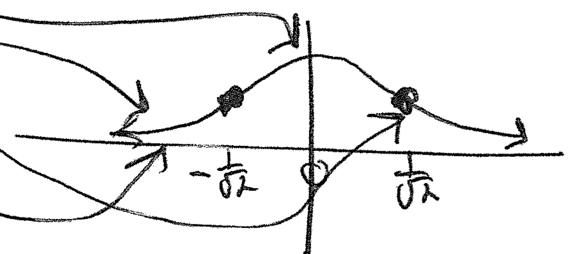
$= \frac{1}{1+0+1} = \frac{1}{2}$

(Oddly, no sin x or cos x appeared!)

20. Let $f(x) = e^{(-x^2)}$. Which of the following are true statements?

(a) f is always concave up
(b) f is always concave down
(c) f is always increasing
(d) f is always descreasing
(e) f is always positive ✓
(f) f is always negative
(g) f has exactly ~~one~~ inflection point
(h) none of the above 2 I.P.

$f'(x) = -2xe^{-x^2}$ $\quad f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$



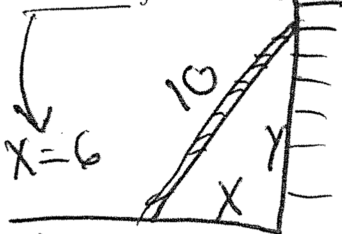
$-2xe^{-x^2} = 0$
$\Downarrow$
Critical: $x = 0$

Possible inflection:
$e^{-x^2}(-2+4x^2) = 0$
$\Downarrow$
$x = \pm\frac{1}{\sqrt{2}}$

Concavity does change at both

Written Problem. Clearly show all steps to receive full credit.

21. (a) A ladder 10 feet long is leaning against a wall. The foot of the ladder is moving away from the wall at a rate of 2 feet per second. How fast is the top of the ladder falling when the foot is six feet away from the wall?



$x = 6$

$\frac{dx}{dt} = 2$

$\frac{dy}{dt}$

$x^2 + y^2 = 10$

$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(10)$

$\cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt} = 0$

When $x = 6$, $y = 8$

$6 \cdot 2 + 8 \cdot \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-12}{8} = -\frac{3}{2}\frac{ft}{sec}$ (so falling at $\frac{3}{2}$ ft/sec)

(b) The volume of an expanding sphere is increasing at a rate of 10 cubic feet per second. How fast is the surface area increasing when the volume is 36π cubic feet?

Find $\frac{dS}{dt}$

Let S = surface area, V = volume, t = time, r = radius

Initially there are four variables, so we need to think of how to handle this complication. One way: eliminate r.

$S = 4\pi r^2$

$V = \frac{4}{3}\pi r^3 \to r^3 = \frac{3}{4\pi}V \to r = \sqrt[3]{\frac{3V}{4\pi}}$

So $S = (4\pi)^1 \frac{(3V)^{2/3}}{(4\pi)^{2/3}} = (4\pi)^{1/3}(3V)^{2/3}$

OR $S = (36\pi)^{1/3} V^{2/3}$

$\frac{dS}{dt} = (36\pi)^{1/3}\frac{2}{3}V^{-1/3}\frac{dV}{dt}$

$\frac{dS}{dt} = \cancel{(36\pi)^{1/3}}\frac{2}{3}\cancel{(36\pi)^{-1/3}} \cdot 10$

$\frac{dS}{dt} = \frac{20}{3}\frac{ft^2}{sec}$

Written Problem. Clearly show all steps to receive full credit.

22. Sketch the graph of a function that has the described characteristics.

    (a) $f(x) > 0$ for all x, $f'(x) < 0$ for all x, and $f''(x) > 0$ for all x

    (b) $f(x) > 0$ for all x, $f'(x) > 0$ for all x, and $f''(x) > 0$ for all x

    (c) $f(x) > 0$ for all x, $f'(x) > 0$ for all negative x, $f'(x) < 0$ for all positive x, and $f''(x) < 0$ for all x

Written Problem. Clearly show all steps to receive full credit.

23. Answer the following questions for $f(x) = 36x + 3x^2 - 3x^3$.

(a) Find the intervals of increase or decrease.

$f'(x) = 36 + 6x - 9x^2$

$3(12 + 2x - 3x^2) = 0 \rightarrow$ Critical numbers $x = \frac{-2 \pm \sqrt{4+144}}{-6}$

$x = \frac{1}{3} \pm \frac{\sqrt{37}}{3} \approx -1.7, 2.4$

$\boxed{\sim -\frac{7}{3}, -\frac{5}{3}}$

$(-\infty, -1.7)$ dec $(-1.7, 2.4)$ inc $(2.4, \infty)$ dec

(b) Find the local maximum and minimum values.

local min value ≈ -38 at -1.7

local max value ≈ 62 at 2.4

(c) Find the intervals of concavity and any inflection points.

$f''(x) = 6 - 18x \rightarrow 6 = 18x \Rightarrow \boxed{x = \frac{1}{3}}$

Concavity changes from up to down, so inflection

$(-\infty, \frac{1}{3})$ concave up $(\frac{1}{3}, \infty)$ concave down